

# CLASSIFICATION OF JOINT NUMERICAL RANGES OF THREE HERMITIAN MATRICES OF SIZE THREE

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**ABSTRACT.** The joint numerical range of three hermitian matrices of order three is a convex and compact subset  $W \subset \mathbb{R}^3$  which is an image of the unit sphere  $S^5 \subset \mathbb{C}^3$  under the hermitian form defined by the three matrices. We label classes of the analyzed set  $W$  by pairs of numbers counting the exposed faces of dimension one and two. Generically,  $W$  belongs to the class of ovals. Assuming  $\dim(W) = 3$ , the faces of dimension two are ellipses and only ten classes exist. We identify an object in each class and use random matrices and dual varieties for illustrations.

## 1. INTRODUCTION

We denote the space of complex  $d$ -by- $d$  matrices by  $M_d$ ,  $d \in \mathbb{N}$ ,  $\mathbb{N} = \{1, 2, 3, \dots\}$ , we write  $M_d^h := \{a \in M_d \mid a^* = a\}$  for the real subspace of hermitian matrices and  $1_d$  for the identity matrix. We write  $\langle x, y \rangle := \overline{x_1}y_1 + \dots + \overline{x_d}y_d$ ,  $x, y \in \mathbb{C}^d$  for the inner product on  $\mathbb{C}^d$  and  $S\mathbb{C}^d := \{x \in \mathbb{C}^d \mid \langle x, x \rangle = 1\}$  for the unit sphere. The *joint numerical range* of a sequence  $F := (F_1, \dots, F_n) \in (M_d^h)^n$ ,  $n \in \mathbb{N}$ , is

$$W(F) := \{\langle x, Fx \rangle \mid x \in S\mathbb{C}^d\} \subset \mathbb{R}^n$$

where  $\langle x, Fx \rangle := \langle x, F_i x \rangle_{i=1}^n$ . For  $n = 2$ , identifying  $\mathbb{R}^2 \cong \mathbb{C}$ , the set  $W(F_1, F_2)$  is known as the *numerical range*  $\{\langle x, Ax \rangle \mid x \in S\mathbb{C}^d\}$  of the matrix  $A := F_1 + iF_2$ . Surprisingly,  $W(F_1, F_2)$  is convex, a fact which is known as the Toeplitz-Hausdorff theorem [50, 23]. Similarly, the joint numerical range  $W(F_1, F_2, F_3)$  is convex [2] for all  $d \geq 3$ . However,  $n = 2, 3$  are the exception with respect to convexity of  $W(F)$  which is in general not convex for  $n \geq 4$ , see [39, 36, 21] for further studies.

The joint numerical range  $W(F)$  is interesting in quantum mechanics, as we recall in Section 2, since for arbitrary  $d, n \in \mathbb{N}$  the convex hull of  $W(F)$  is the image of a linear map  $\mathcal{M}_d \rightarrow L(F)$  on the *state space* of the algebra  $M_d$ , that is on the set  $\mathcal{M}_d$  of positive semi-definite matrices of trace one,  $L(F) = \text{conv } W(F)$ . So, in the case  $d = n = 3$  we have

$$L(F) = W(F).$$

We call  $L(F)$  *convex support* [51] by its name in statistics [4]. For triples  $F$  of hermitian 3-by-3 matrices, one of us ran Wolfram Mathematica to compute the joint numerical range from random matrices [55], and printed them on a 3D-printer. The printout shown in Figure 1f) was the starting point for this research.

The partition of  $(M_d^h)^n$  produced by our classification ( $d = n = 3$ ) will be coarser than the classes of the equivalence relation where  $F = (F_1, \dots, F_n) \in (M_d^h)^n$  is equivalent to

$$(1.1) \quad \begin{aligned} &\bullet U^*FU = (U^*F_1U, \dots, U^*F_nU) \text{ for every unitary } U \in M_d, \\ &\bullet F' \in (M_d^h)^n \text{ if the span of } 1_d, F_1, \dots, F_n \text{ is the span of } 1_d, F'_1, \dots, F'_n. \end{aligned}$$

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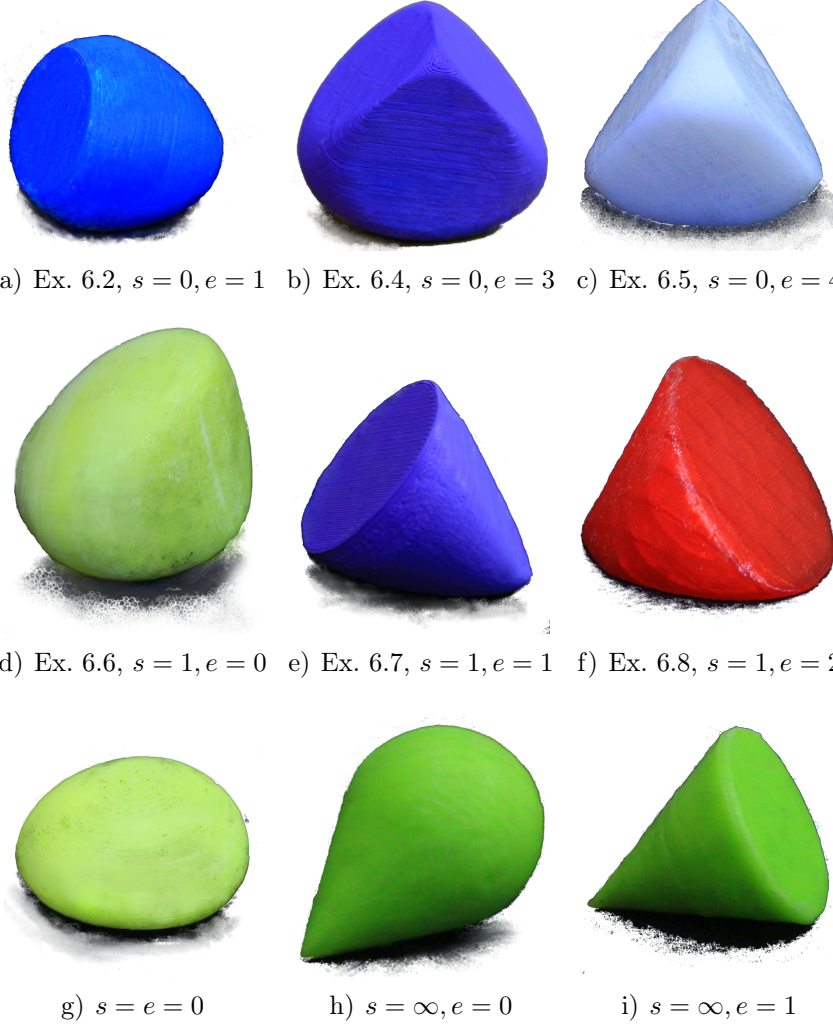


FIGURE 1. 3D printouts of exemplary joint numerical ranges of triples of hermitian matrices of size three:  $s$  denotes the number of segments,  $e$  the number of ellipses in the boundary

The first item of (1.1) is motivated by unitary invariance of  $W(F)$ , the second by equivariance under invertible affine transformations of  $\mathbb{R}^n$ .

The elliptical range theorem [35] states that if  $d = n = 2$  then  $W(F)$  is a singleton, segment, or filled ellipse. For arbitrary  $d$  and  $n = 2$  we are aware of two classifications of  $W(F)$ . First, let  $d, n$  be arbitrary. We say that  $F$  is *unitarily reducible* if there exists a unitary  $U \in M_d$  such that the matrices  $U^*FU$  have a common block diagonal form with at least two proper blocks. Otherwise  $F$  is *unitarily irreducible*. Returning to  $n = 2$ , one classification of  $W(F)$  is derived [29] from a plane algebraic curve whose convex hull is  $W(F)$ : For  $d = 3$  the numerical range is a singleton, segment, triangle, ellipse, or the convex hull of an ellipse and a point outside the ellipse, if  $F$  is unitarily reducible; it is an ellipse, the convex hull of a quartic curve (with a flat portion on the boundary), or the convex hull of a sextic curve (an oval), if  $F$  is unitarily irreducible. The sets of matrices where  $W(F)$  has any of these shapes are characterized in terms of eigenvalues and matrix entries [30, 44, 42], their closures









$s = 1$					
$s = 0$					
	$e = 0$	$e = 1$	$e = 2$	$e = 3$	$e = 4$

TABLE 1. Equivalence classes of  $L(F)$  without corner points. Each of the segments in  $\mathcal{L}(F)$  is represented by a segment, each of the ellipses by a circle. Dots denote intersection points between large faces.

by further spectral data [49]. Another classification, for  $n = 2$  is based on a convex duality of  $W(F)$  to a so-called rigidly convex set [24, 25].

Despite an increasing interest in operator theory [7, 31, 12, 13, 11], a classification of  $L(F)$  is unknown for  $n \geq 3$  even in the simplest case of 3-by-3 matrices. The main technical focus of the present work is the notion of *exposed face* of  $L(F)$ , which can be the empty set or the set of maximizers of any linear functional on  $L(F)$ . Intuitively, we think of a non-empty exposed face as a 'flat boundary portion', if it is neither equal to  $L(F)$ , nor to a singleton. For  $d = 3$ , Lemma 4.2 shows that these exposed faces are necessarily segments, filled ellipses, or filled ellipsoids. We call them *large faces* of  $L(F)$  and collect them in the set ( $d = 3$ ,  $n \in \mathbb{N}$ )

$$(1.2) \quad \mathcal{L}(F) := \{G \mid G \neq L(F) \text{ is an exposed face of } L(F) \text{ which is a segment, a filled ellipse, or a filled ellipsoid}\}.$$

An analysis [12] of singularities of the projective algebraic set

$$(1.3) \quad S_F := \{(u_0 : \dots : u_n) \in \mathbb{P}\mathbb{C}^{n+1} \mid \det(u_0 \mathbf{1} + u_1 F_1 + \dots + u_n F_n) = 0\}$$

shows that the number of ellipses in  $\mathcal{L}(F)$  is at most four, if  $d = n = 3$ . This is also a corollary of our classification. In Section 6 we shall see examples of a certain dual variety of  $S_F$  which contains (unbounded) lines. The first such example was found in [13] and proves that the result of [29] (the numerical range is the convex hull of a plane algebraic curve) cannot be generalized directly to higher dimensions.

We prove in Theorem 4.3 that  $L(F)$  is generically an oval for  $d \geq 2$ ,  $n \leq 3$ , that is  $L(F)$  is a smooth, compact, and convex set with an interior point. This follows from the von Neumann-Wigner *non-crossing rule* [38, 21] and from a study of normal cones in Section 3. We prove that  $L(F)$  is an oval, if  $\mathcal{L}(F) = \emptyset$  and  $\dim(L(F)) = n$  (the converse is immediate). On the other hand, the *crossing rule* [19] and Lemma 4.4 about pre-images of  $\mathcal{M}_3 \rightarrow L(F)$  show in Remark 4.5 that ovals are the exception for  $d = 3$ ,  $n \geq 4$ . For real symmetric matrices  $F$ , ovals are generic for  $d \geq 2$ ,  $n \leq 2$  and exceptional for  $d = 3$  already when  $n \geq 3$ . Section 4 also collects other methods to study exposed faces, such as spectral representation, discriminants, sum of squares decompositions [32, 27], and *corner points*, that is points which lie on  $n$  supporting hyperplanes with linearly independent normal vectors.

We now derive the classification of the joint numerical range for  $d = n = 3$ , using the results of Section 5. Let  $s$  denote the number of *segments* and  $e$  of *ellipses* in  $\mathcal{L}(F)$ . By projecting  $L(F)$  onto planes, the classification of the numerical range of

a 3-by-3 matrix shows that large faces intersect mutually (Lemma 5.1). Further, we assume that  $L(F)$  has no corner point. Then, no point lies on three mutually distinct large faces (Lemma 5.2) so Theorem 5.3 shows that the union of large faces contains an embedded complete graph with one vertex on each large face. Now, a theorem about graph embedding [43] shows  $s + e \leq 4$ . Lemma 5.4 shows  $s = 0, 1$  (namely  $L(F)$  has a corner point, if  $s \geq 2$ ). Observing that  $s = 1$  implies that the vertex degree is at most two, we arrive at the eight classes of Table 1.

For two-dimensional  $L(F)$  we have  $e = 0$ . By projecting to a plane,  $L(F)$  corresponds to the numerical range of a 3-by-3 matrix whose classification is cited above [29]. Notice that  $L(F)$  belongs to one of four classes of 2D objects characterized by the number of segments  $s = 0, 1, 2, 3$ . For  $s = 0$  the numerical range is the convex hull of an ellipse or of a sextic curve. For  $s = 1$  it is the convex hull of a quartic curve, for  $s = 2$  the convex hull of an ellipse and a point outside the ellipse, and for  $s = 3$  a triangle. We see, in the restriction to planar sets, our classification is coarser than that of [29].

There are ten classes of three-dimensional  $L(F)$ . All objects with corner points belong to class  $(\infty, 0)$  or  $(\infty, 1)$  by Lemma 4.7. Examples are depicted in Figure 1h) and 1i), respectively. All other objects belong to one of the eight classes of Table 1, as we have seen in the penultimate paragraph. In Section 6 we discuss an example of each class, except for the class  $(0, 0)$  of ovals which were studied in Section 4 of [31]. An example of  $(1, 1)$  was published in [13], one of  $(1, 2)$  in [9], and one of  $(0, 4)$  in [26]. We are unaware of examples in the literature for

$$(s, e) = (0, 1), (0, 2), (0, 3), \text{ or } (1, 0).$$

The classes of  $(1, e)$ ,  $e = 0, 1, 2$ , where  $\mathcal{L}(F)$  contains a segment, answer a question of Remark 5.9 of [53] (already answered in Example 4.2 of [45]). Namely, a limit of extreme points of  $\mathcal{M}_d$ ,  $d \in \mathbb{N}$ , is again an extreme point. The question whether the analogue property holds for projections of  $\mathcal{M}_d$  is answered negative by any object of class  $(1, e)$ . In fact, any point in the relative interior of the segment of  $\mathcal{L}(F)$  is a limit point of extreme points but not an extreme point itself.

Finally, we mention that the graph embedding method fails for  $n > 3$  because any graph can be embedded into the three-sphere  $S^3$ . Still, for  $d = 3$  all large faces of  $L(F)$  intersect pairwise.

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## 2. QUANTUM STATES

Our interest in the joint numerical range is its role in quantum mechanics where the hermitian matrices  $M_d^h$  are called Hamiltonians, energies, or observables, see e.g. [6], Secs 5.1 and 5.2, and they correspond to measurable physical quantities having  $d$  energy levels.

Usually a (complex)  $C^*$ -subalgebra  $\mathcal{A}$  of  $M_d$  is considered as the algebra of observables [1]. If  $a \in M_d$  is positive semi-definite then we write  $a \succeq 0$ . The physical state of a quantum system is given by a linear functional  $f : \mathcal{A} \rightarrow \mathbb{C}$  which is normalized ( $f(1) = 1$ ) and positive, that is  $f(a) \geq 0$  for  $a \succeq 0$ ,  $a \in \mathcal{A}$ , called a *state* on  $\mathcal{A}$  in the mathematical terminology. Using the inner product  $\langle a, b \rangle := \text{tr}(a^*b)$ ,  $a, b \in M_d$ , there exists a unique *density matrix*  $\rho \in \mathcal{A}$  such that  $f(a) = \text{tr}(\rho a)$ ,  $a \in \mathcal{A}$ , where  $\rho \succeq 0$  has unit trace  $\text{tr}(\rho) = 1$ , see e.g. [1], Sec. 4. We use the notions of state and density matrix synonymously and we denote the *state space* of  $\mathcal{A}$  by

$$\mathcal{M}(\mathcal{A}) := \{\rho \in \mathcal{A} \mid \rho \succeq 0, \text{tr}(\rho) = 1\}.$$

It is well-known that  $\mathcal{M}(\mathcal{A})$  is a compact convex subset of  $M_d^h$ , see for example Theorem 4.6 of [1]. We are mainly interested in  $\mathcal{M}_d := \mathcal{M}(M_d)$  but in Sec. 4 also in the compressed algebra  $pM_dp$  where  $p \in M_d$  is an orthogonal projection operator, that is  $p^2 = p^* = p$ . The state space  $\mathcal{M}(pM_dp)$  is, as we recall in Sec. 4, an exposed face of  $\mathcal{M}_d$ , see [1, 51]. The state space  $\mathcal{M}_2$  is a Euclidean ball, called *Bloch ball*, but  $\mathcal{M}_d$  is not a ball [6] for  $d \geq 3$ . Although several attempts were made to analyze properties of this set [28, 5, 47, 33], its complicated structure requires further studies.

In physics, the expected value of a Hamiltonian  $a \in M_d$ , if a quantum system is in the state  $\rho \in \mathcal{M}_d$ , is the real number  $\langle \rho, a \rangle$ . Thus the standard numerical range  $W(F_1, F_2)$  of a non-hermitian operator  $A = F_1 + iF_2$  can be considered as the set of all possible outcomes of measurements of two hermitian operators  $F_1$  and  $F_2$  performed on two copies of the same quantum state  $\rho$ . Furthermore, this set can be identified [16, 40] as a projection of the set  $\mathcal{M}_d$  of quantum states onto a two-plane.

In this work we analyze numerical range of a triple of hermitian matrices of order three. As in the case of a pair of matrices, they can be interpreted as sets of possible results of measurements of three hermitian observables performed on three copies of the same quantum state. On the other hand they form projections of the 8D set of density matrices of order three into  $\mathbb{R}^3$ , see [22]. A special case of the latter are projections to subsystems, whose geometry was recently studied [54, 10] to investigate the structure of reduced density matrices for complex many-body quantum systems.

Consider a general case,  $d, n \in \mathbb{N}$ , of joint numerical range of  $n$  hermitian matrices of arbitrary size  $d$ . In this paper we will use the following linear map

$$(2.1) \quad \mathbb{E} : M_d^h \rightarrow \mathbb{R}^n, \quad a \mapsto (\langle a, F_1 \rangle, \dots, \langle a, F_n \rangle).$$

The set of expected values of  $F = (F_1, \dots, F_n)$  is

$$L(F) := \mathbb{E}(\mathcal{M}_d) = \{\mathbb{E}(\rho) \mid \rho \in \mathcal{M}_d\}.$$

We call the set  $L(F)$  *convex support* [51] following statistics language usage [4]. It is a compact convex subset of  $\mathbb{R}^n$ . The convex support equals the convex hull of the joint numerical range

$$(2.2) \quad L(F) = \text{conv}(W(F)).$$

Recall that for  $n = 3$  and  $d \geq 3$  we have  $L(F) = W(F)$  because  $W(F)$  is convex.

A proof of equation (2.2) is given in [22] using linear algebra. Because of its importance we give a second, convex geometric, proof. A *face* of a convex subset  $C \subset \mathbb{R}^m$  is a convex subset of  $C$  which contains the endpoints of every open segment in  $C$  which it intersects. The faces  $\emptyset, C$  are *improper faces* of  $C$ , all other faces are *proper faces*. An *exposed face* of  $C$  is defined as the set of maximizers in  $C$  of some

linear functional;  $\emptyset$  is an exposed face by definition. It is easy to prove that every exposed face is a face. If a singleton is a face or exposed face then we call its element an *extreme point* or *exposed point*, respectively.

*Minkowski's theorem* affirms that every compact convex subset of  $\mathbb{R}^m$ ,  $m \in \mathbb{N}$ , is the convex hull of its extreme points, see for example Corollary 1.4.5 of [48].

*Proof of (2.2):* For every unit vector  $x \in S\mathbb{C}^d$ , the orthogonal projection onto the span of  $x$ , in Dirac's notation  $|x\rangle\langle x|$ , lies in  $\mathcal{M}_d$ . Since  $\langle x, ax \rangle = \langle |x\rangle\langle x|, a \rangle$  holds for all  $a \in \mathcal{M}_d$ , this shows  $W(F) \subset L(F)$  and implies  $\text{conv}(W(F)) \subset L(F)$  because  $L(F)$  is convex.

Conversely, using Minkowski's theorem, it suffices to show that every extreme point  $p$  of  $L(F)$  lies in  $W(F)$ . It is easy to show that the pre-image of  $p$  under the linear map  $\rho \mapsto \langle \rho, F \rangle$  restricted to  $\mathcal{M}_d$  is a face of  $\mathcal{M}_d$ . This face is a non-empty compact and convex set, so it contains by Minkowski's theorem an extreme point  $\rho$ . Clearly  $\rho$  is an extreme point of  $\mathcal{M}_d$  and, since the extreme points of  $\mathcal{M}_d$  are the rank-one orthogonal projection operators (see for example Proposition 4.1 of [1]), the state has the form  $\rho = |x\rangle\langle x|$  for a unit vector  $x \in S\mathbb{C}^d$ . Now  $p = \langle \rho, F \rangle = \langle x, Fx \rangle \in W(F)$  completes the proof.  $\square$

### 3. NORMAL CONES AND OVALS

We prove a characterization of ovals for the family of convex sets where every extreme ray of a normal cone is a normal cone itself. We recall that convex support sets belong to this family of convex sets.

Let  $C \subset \mathbb{R}^m$ ,  $m \in \mathbb{N}$ , be a compact convex subset. The *normal cone* of  $C$  at  $x \in C$  is  $N(x) := \{u \in \mathbb{R}^m \mid \forall y \in C : \langle u, y - x \rangle \leq 0\}$ . The non-zero elements of  $N(x)$  are known as outer normal vectors of  $C$  at  $x$ . The point  $x$  is *smooth* if it has a unique outer unit normal vector. We call  $x$  a *corner point* [17] if  $\dim(N(x)) = m$ . We define an *oval* as a non-empty, convex, and compact subset of  $\mathbb{R}^m$  each of whose boundary points is a smooth exposed point. The normal cone  $N(G) := N(x)$  of a non-empty face  $G$  of  $C$  is well-defined as the normal cone of any point  $x$  in the *relative interior* of  $G$ , that is the interior of  $G$  with respect to the topology of the affine hull of  $G$ . See for example Section 4 of [52] or Section 2.2 of [48] about the consistency of this definition and set  $N(\emptyset) := \mathbb{R}^m$ . A *ray* is a set  $\{\lambda u \mid \lambda \geq 0\} \subset \mathbb{R}^m$  where  $u \in \mathbb{R}^m$  and  $u \neq 0$ . An *extreme ray* of a convex set is a ray which is a face of the given set.

What makes normal cones of  $L(F)$  special is that all their non-empty faces are normal cones, too. For two-dimensional convex sets  $C$  in  $\mathbb{R}^2$  this means that a boundary point of  $C$  is smooth unless it is the intersection of two one-dimensional faces of  $C$ , see Section 1.2.5 of [52]. An example where this property fails is the truncated disk  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, x \leq \frac{1}{2}\}$ . The normal cone at  $\frac{1}{2}(1, \sqrt{3})$  has two extreme rays but only one of them, the non-negative  $x$ -axis, is a normal cone.

We denote the set of exposed faces and normal cones of  $C$  by  $\mathcal{E}_C$  and  $\mathcal{N}_C$ , respectively. Each of these sets is a *lattice* partially ordered by inclusion, that is the infimum  $x \wedge y$  and supremum  $x \vee y$  exist for all  $x, y$  in the set. A *chain* in a lattice is a totally ordered subset, its *length* is its cardinality minus one. The *length* of a lattice is the supremum of the lengths of all its chains. Lattices of faces have been studied earlier [3, 37], in particular these of state spaces [1], and linear images  $L(F)$

of state spaces [51]. By Proposition 4.7 of [52], if  $C$  is not a singleton then

$$(3.1) \quad N : \mathcal{E}_C \rightarrow \mathcal{N}_C$$

is an antitone lattice isomorphism. This means that  $N$  is a bijection and for all exposed faces  $G, H$  we have  $G \subset H$  if and only if  $N(G) \supset N(H)$ .

Ovals are smooth, unlike for example the intersection of two disks  $\{(x, y) \in \mathbb{R}^2 \mid (x \pm \frac{1}{2})^2 + y^2 \leq 1\}$ . Still, all proper exposed faces of this example are singletons. This suffices to characterize ovals within the following class of convex sets.

**Theorem 3.1.** *Every oval in  $\mathbb{R}^m$  has dimension  $m$ . Let  $C \subset \mathbb{R}^m$  be a convex compact subset,  $\dim(C) = m$ , such that every extreme ray of every normal cone of  $C$  is a normal cone of  $C$ . Then  $C$  is an oval if and only if all proper exposed faces of  $C$  are singletons.*

*Proof:* If  $C$  is an oval, then by definition its boundary is covered by singleton faces. Since  $C$  is the disjoint union of the relative interiors of all its faces, see for example Theorem 2.1.2 of [48], only  $C$  may be no singleton. In particular, all proper exposed faces are singletons. By contradiction, if  $\dim(C) < m$  then  $C$  lies in its own boundary and is thus a singleton, having at least two outer normal unit vectors.

Conversely, we assume that all proper exposed faces of  $C$  are singletons. Since every proper face lies in a proper exposed face, see for example Lemma 4.6 of [52], it follows that all proper faces are exposed points. By assumption  $\dim(C) = m$ , so the proper faces, and hence the exposed points, cover the boundary of  $C$ .

We prove that  $C$  is smooth. By assumption, the lattice  $\mathcal{E}_C$  of exposed faces has length two. The isomorphism (3.1) shows that the lattice  $\mathcal{N}_C$  of normal cones has length two. Let  $x$  be a boundary point of  $C$ . As  $\dim(C) = m$ , the normal cone  $N(x)$  contains no line so it has at least one extreme ray, denoted by  $r$  (see e.g. Theorem 1.4.3 of [48]). By assumptions, the ray  $r$  is a normal cone of  $C$ . So

$$\{0\} \subset r \subset N(x) \subset \mathbb{R}^m$$

proves  $N(x) = r$ , for otherwise  $\mathcal{N}_C$  had length three. Therefore  $x$  has a unique outer unit normal vector.  $\square$

Theorem 3.1 applies to convex support sets by the following fact.

**Fact 3.2.** For all  $n, d \in \mathbb{N}$  and  $F \in (M_d^h)^n$  the intersection of any two normal cones  $\neq \mathbb{R}^n$  of  $L(F)$  is a non-empty face of each of them. Every non-empty face of every normal cone of  $L(F)$  is a normal cone of  $L(F)$ .

The first assertion of Fact 3.2 is proved in Proposition 4.8 of [52]. We give two proofs for the second one. The first one follows Section 1.3 of [52].

*1st Proof of Fact 3.2:* Proposition 2.9 and 2.11 of [51] prove for every  $u \in \mathbb{R}^n \setminus \{0\}$  and for every state  $\rho$  in the relative interior of the exposed face  $\mathbb{F}_{\mathcal{M}_d}(F(u))$  that  $F(u)$  lies in the relative interior of the normal cone of  $\mathcal{M}_d$  at  $\rho$ . Hence, Proposition 7.4 and 7.6 of [52] show that every non-empty face of a normal cone of  $\mathcal{M}_d$  is a normal cone of  $\mathcal{M}_d$ . Corollary 7.7 of [52] proves that  $L(F)$ , being a projection of  $\mathcal{M}_d$ , has the analogue property, stated in Fact 3.2.  $\square$

Second we use a polarity to a spectrahedron. The *polar* of a subset  $C \subset \mathbb{R}^m$  is the closed convex set  $C^\circ := \{u \in \mathbb{R}^m \mid \forall x \in C : \langle x, u \rangle \leq 1\}$ .

*2nd Proof of Fact 3.2:* Assuming that zero is an interior point of  $L(F)$ , the positive hull operator restricts to a lattice isomorphism from the faces of  $L(F)^\circ$  onto the non-empty faces of normal cones of  $L(F)$ , see Theorem 8.3 of [52], and further to a lattice isomorphism from the exposed faces of  $L(F)^\circ$  onto the normal cones of  $L(F)$ , see equation (31) of [52]. The set  $L(F)^\circ$  is a *spectrahedron*, see Section 3 of [41], Section 2 of [12], or Remark 2.17(3) of [51] for definitions and proofs, and the claim follows then because all faces of a spectrahedron are exposed [41].  $\square$

#### 4. EXPOSED FACES

We study exposed faces of the convex support  $L(F)$  with a focus on large faces, ovals, and corner points [17], which are also known as *conical points* [7].

We recall the spectral representation of exposed faces and introduce discriminants and their sum of squares decomposition to study eigenvalue degeneracy. We analyze proper faces of  $L(F)$  for 3-by-3 matrices ( $d = 3$ ). Further, we prove that the convex support is generically an oval for at most three hermitian matrices ( $d \in \mathbb{N}, n \leq 3$ ). We discuss pre-images of exposed points under  $\mathcal{M}_3 \rightarrow L(F)$  and show that the convex support is generically not an oval for  $d = 3$  and  $n \geq 4$  — for real symmetric matrices already when  $d = 3$  and  $n \geq 3$ . For  $d = 3$  we describe  $L(F)$  which has a corner point.

Unless otherwise specified,  $d, n \in \mathbb{N}$  are arbitrary in this section, and as before  $F = (F_1, \dots, F_n) \in (M_d^h)^n$ . Exposed faces transform very simple under linear maps, see for example Lemma 5.4 of [52]. In formulae, the exposed face of  $L(F)$

$$\mathbb{F}_{L(F)}(u) := \operatorname{argmax}_{x \in L(F)} \langle x, u \rangle, \quad u \in \mathbb{R}^n,$$

has a pre-image in the state space  $\mathcal{M}_d$  under (2.1) equal to

$$(4.1) \quad \mathbb{E}_{\mathcal{M}_d}^{-1}(\mathbb{F}_{L(F)}(u)) = \mathbb{F}_{\mathcal{M}_d}(F(u)) := \operatorname{argmax}_{\rho \in \mathcal{M}_d} \langle \rho, F(u) \rangle$$

where

$$F(u) := u_1 F_1 + \dots + u_n F_n.$$

An algebraic aspect appears because the exposed face  $\mathbb{F}_{\mathcal{M}_d}(a)$  of  $\mathcal{M}_d$ , for  $a \in M_d^h$ , is  $\mathbb{F}_{\mathcal{M}_d}(a) = \mathcal{M}(pM_dp)$  where  $p$  is the spectral projection of  $a$  corresponding to the maximal eigenvalue of  $a$ , see [1] or [51]. Therefore, (4.1) shows

$$(4.2) \quad \mathbb{E}_{\mathcal{M}_d}^{-1}(\mathbb{F}_{L(F)}(u)) = \mathcal{M}(pM_dp), \quad u \in \mathbb{R}^n,$$

where  $p$  is the spectral projection of  $F(u)$  corresponding to the maximal eigenvalue of  $F(u)$ .

**Remark 4.1** (Spectral representation of faces). A proof is given in Section 3.2 of [21] that  $\max\{\langle x, u \rangle \mid x \in W(F)\}$  is the maximal eigenvalue of  $F(u)$ ,  $u \in \mathbb{R}^n$ . This result goes back to Toeplitz [50] for  $n = 2$  and follows also from (2.2) and (4.2).

Towards an analysis of the set of large faces  $\mathcal{L}(F)$ , defined in (1.2), equation (4.2) shows that for such a face to exist it is necessary that  $F(u)$  has a double eigenvalue for some  $u \in \mathbb{R}^n \setminus \{0\}$ . This can be checked using discriminants and their sum of squares decomposition.



Let  $a_1, a_2, a_3 \in \mathbb{R}$  and consider the monic polynomial  $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$  of degree three. If  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$  denote its roots then the *discriminant* is defined by

$$\prod_{1 \leq i < j \leq 3} (\lambda_i - \lambda_j)^2,$$

which is  $-27a_3^2 + 18a_1a_2a_3 - 4a_1^3a_3 - 4a_2^3 + a_1^2a_2^2$ , as we see for example by comparing coefficients. The *discriminant*  $\delta(A)$  of a 3-by-3 matrix  $A$  is the discriminant of the characteristic polynomial  $\det(A - \lambda\mathbb{1})$ . So,  $A$  has a multiple eigenvalue if and only if  $\delta(A) = 0$ .

If  $A$  is a normal 3-by-3 matrix, let  $v_0$  be the column vector with the nine entries of the identity matrix  $\mathbb{1}_3$ . Let  $v_1$  be the column vector with the entries of  $A$  in the same index ordering used for  $v_0$ , and  $v_2$  the column vector with the entries of  $A^2$  again in the same ordering. We write the vectors into a 9-by-3 matrix  $(v_0, v_1, v_2)$ . For a normal 3-by-3 matrix  $A$  we have [27]

$$(4.3) \quad \delta(A) = \sum_{\nu} |M_{\nu}|^2$$

where the sum extends over the 84 three-element subsets  $\nu$  of  $\{1, 2, 3\} \times \{1, 2, 3\}$  and  $M_{\nu}$  is the minor of  $(v_0, v_1, v_2)$  corresponding to the submatrix with rows  $\nu$ . It is worth noting that the discriminant of a real symmetric 3-by-3 matrix can be decomposed into five squares [15].

Large faces of  $L(F)$  were introduced in (1.2) for  $d = 3$ . Let  $G$  be one of them, that is, let  $u \in \mathbb{R}^n$ ,  $G := \mathbb{F}_{L(F)}(u)$ ,  $\dim(G) \geq 1$  and  $G \neq L(F)$ . By (4.2), the pre-image of  $G$  under  $\mathbb{E}|_{\mathcal{M}_3} : \mathcal{M}_3 \rightarrow L(F)$  has the form of  $\mathcal{M}(pM_3p)$  where  $p$  is an orthogonal projection operator. By assumption,  $G$  is a proper face and no singleton, so  $p$  has rank two. Thus,  $\mathcal{M}(pM_3p)$  is a three-dimensional Euclidean ball and  $G$  is a segment, a filled ellipse, or a filled ellipsoid.

Before summarizing the preceding paragraph, we recall that an *atom* in a lattice  $(\mathcal{L}, \leq, \wedge, \vee, 0)$  with smallest element 0 is an element  $x \in \mathcal{L}$ ,  $x \neq 0$ , such that  $y \leq x$ ,  $y \neq x$  implies  $y = 0$  for all  $y \in \mathcal{L}$ . Similarly, a *coatom* in a lattice  $(\mathcal{L}, \leq, \wedge, \vee, 1)$  with greatest element 1 is an element  $x \in \mathcal{L}$ ,  $x \neq 1$ , such that  $y \geq x$ ,  $y \neq x$  implies  $y = 1$  for all  $y \in \mathcal{L}$ .

**Lemma 4.2.** *Let  $F$  be a sequence of  $n \in \mathbb{N}$  hermitian 3-by-3 matrices. Every proper face of  $L(F)$  which is not a singleton is a segment, a filled ellipse, or a filled ellipsoid. Moreover, that face is an exposed face of  $L(F)$  and a coatom of  $\mathcal{E}_{L(F)}$ .*

*Proof:* Every proper face  $G$  of  $L(F)$  lies in a proper exposed face (see for example Lemma 4.6 of [52]) which is a segment, ellipse, or ellipsoid, as mentioned above. Since  $G$  is no singleton, it is equal to that proper exposed face. It is a coatom of the lattice  $\mathcal{E}_{L(F)}$  of exposed faces since ellipses and ellipsoids have no segments or ellipses on the relative boundary.  $\square$

The generic convex support of at most three hermitian matrices is an oval.

**Theorem 4.3.** *Let  $n \in \{1, 2, 3\}$  and  $d \geq 2$ . Then the set of  $n$ -tuples of hermitian  $d$ -by- $d$  matrices  $F \in (M_d^{\mathbb{h}})^n$  such that  $L(F)$  is an oval is open and dense in  $(M_d^{\mathbb{h}})^n$ .*

*Proof:* For  $n = 1, 2, 3$  and  $d \in \mathbb{N}$  the set  $\mathcal{O}_1$  of all  $F \in (M_d^{\mathbb{h}})^n$  where every matrix in the pencil  $\{F(u) \mid u \in \mathbb{R}^n \setminus \{0\}\}$  has  $d$  simple eigenvalues is open and dense in  $(M_d^{\mathbb{h}})^n$ , this was shown in Prop. 4.9 of [21]. Hence, for  $F \in \mathcal{O}_1$  all proper exposed faces of  $L(F)$  are singletons by (4.2). Secondly, since  $n + 1 \leq \dim_{\mathbb{R}}(M_d^{\mathbb{h}}) = d^2$  holds by the assumptions  $n \leq 3$  and  $d \geq 2$ , it is easy to prove that  $\mathbb{1}_d, F_1, \dots, F_n$

are linearly independent for  $F$  in an open and dense subset  $\mathcal{O}_2$  of  $(M_d^h)^n$ , that is  $\dim(L(F)) = n$  holds for  $F \in \mathcal{O}_2$ . The extreme rays of every normal cone of  $L(F)$  are normal cones by Fact 3.2. Hence Theorem 3.1 proves that  $L(F)$  is an oval for all  $F$  in  $\mathcal{O}_1 \cap \mathcal{O}_2$ . The proof is completed by observing that in a metric space the intersection of two open and dense subsets is dense.  $\square$

Conversely, if a double maximal eigenvalue of  $F(u)$  fails to generate a large face of  $L(F)$  then  $L(F)$  has the special shape of a Euclidean ball.

**Lemma 4.4.** *Let  $n, D \in \mathbb{N}$ ,  $F \in (M_3^h)^n$ , and  $\dim(L(F)) = D$ . If the pre-image  $\mathbb{E}_{\mathcal{M}_3}^{-1}(x)$  of an exposed point  $x$  of  $L(F)$  is not a singleton then  $D \leq 4$  and there exists  $\varphi \in \mathbb{R}$  such that  $F$  is equivalent modulo (1.1) to  $F^\varphi := (F_1^\varphi, \dots, F_D^\varphi)$  where*

$$\begin{aligned} F_1^\varphi &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & F_2^\varphi &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ F_3^\varphi &:= \begin{pmatrix} 0 & 0 & \cos(\varphi) \\ 0 & 0 & i \sin(\varphi) \\ \cos(\varphi) & -i \sin(\varphi) & 0 \end{pmatrix}, & \text{and } F_4^\varphi &:= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \end{aligned}$$

If  $D \geq 3$  then  $\varphi$  is unique in  $[0, \pi)$ . The convex support  $L(F^\varphi)$  is the unit ball of  $\mathbb{R}^D$ , and the pre-image of  $(x_1, \dots, x_D) \in S^{D-1}$  is a singleton unless  $x_1 = 1$ . If the matrices  $F$  are real symmetric then necessarily  $D \leq 3$  and  $\varphi = 0 \pmod{\pi}$ .

*Proof:* Let  $x \in L(F)$  be an exposed point, say  $\{x\} = \mathbb{F}_{L(F)}(u_0)$  for a unit vector  $u_0 \in \mathbb{R}^n$ . Applying an orthogonal transformation of  $\mathbb{R}^n$  we take  $u_0 := (1, 0, \dots, 0)$ . By (4.2) we have  $\mathbb{E}_{\mathcal{M}_3}^{-1}(x) = \mathcal{M}(p\mathcal{M}_3p)$  where  $p$  is the spectral projection of  $F(u_0)$  corresponding to the maximal eigenvalue. Since  $\mathbb{E}_{\mathcal{M}_3}^{-1}(x)$  is not a singleton and not equal to  $\mathcal{M}_3$  it follows that  $p$  has rank two. Notice for all  $i = 1, \dots, n$  that  $pF_i p$  is a scalar multiple of  $p$ . This is clear for  $i = 1$  because  $p$  is a spectral projection of  $F_1$ . For  $i \geq 2$  one could otherwise find two points  $\rho_1, \rho_2 \in \mathcal{M}(p\mathcal{M}_3p)$  with  $\langle \rho_1, F_i \rangle \neq \langle \rho_2, F_i \rangle$  but then  $\{x_0\} = \mathbb{F}_{L(F)}(u_0) = \mathbb{E}(\mathcal{M}(p\mathcal{M}_3p))$  could not be a singleton. Applying a unitary similarity to all matrices  $F$  we take  $p = \text{diag}(1, 1, 0)$ . Further transformations from (1.1) then show  $D \leq 4$  and we can choose to have  $F$  equivalent to  $F^\varphi := (F_1^\varphi, \dots, F_D^\varphi)$  with the  $F_i^\varphi$  defined above. Orthogonal similarity can be used if  $F$  consists of real symmetric matrices where  $D \leq 3$  follows and where  $F^\varphi$  is real symmetric. For  $D \geq 3$  it is easy to see that two sequences  $F^\varphi$  are non-equivalent modulo (1.1) for different values of  $\varphi \pmod{\pi}$ .

Let  $n = D \leq 4$ . The support function  $h(u) := \max_{x \in L(F^\varphi)} \langle u, x \rangle$  is the maximal eigenvalue of  $F^\varphi(u)$ , where  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$  (Remark 4.1). For a unit vector  $u$  the matrix  $F^\varphi(u)$  has eigenvalues  $\{-1, u_1, 1\}$  so  $h(u) = 1$  and this shows that  $L(F^\varphi)$  is the unit ball. The pre-image of the exposed face  $\{u_0\} = \mathbb{F}_{L(F^\varphi)}(u_0)$  is a three-dimensional ball because  $F_1^\varphi$  has a double maximal eigenvalue (4.2). To see that  $u_0$  is the unique extreme point having more than one pre-image it suffices to show for  $n \geq 2$  that  $F^\varphi(u)$  has no double eigenvalue for any unit vector  $u$  which is not collinear with  $u_0$ . The latter holds because the discriminant of  $F^\varphi(u)$  is  $\delta(F^\varphi(u)) = 4(u_2^2 + \dots + u_n^2)^2$ .  $\square$

We notice that Lemma 4.4 generalizes a theorem of [34]. Now we can prove for  $n \geq 5$  hermitian matrices of order 3 that the set  $L(F)$  differs generically from an oval (likely also for  $n = 4$ ). For real symmetric matrices of order 3 it differs generically from an oval already when  $n = 4$  (and likely also for  $n = 3$ ).

**Remark 4.5.** Let  $d \neq 0, \pm 1 \pmod{8}$ . If  $n \geq 4$  then Theorem D of [19] shows that for every linear map  $\Psi : \mathbb{R}^n \rightarrow M_d^h$  exists a point  $u \neq 0$  in  $\mathbb{R}^n$  such that the matrix  $\Psi(u)$  has a multiple eigenvalue. Moreover, if the range of  $\Psi$  is replaced by the space of real symmetric  $d$ -by- $d$  matrices then Theorem B of [19] shows that  $n \geq 3$  suffices for the existence of such a point.

A geometric interpretation is straight forward for 3-by-3 matrices  $F \in (M_3^h)^n$ . If  $n \geq 9$  then  $L(F)$  cannot have an interior point and is not an oval. This is clear because  $D \leq \dim(\mathcal{M}_3) = 8$  holds for  $D := \dim(L(F))$ . On the contrary, let  $4 \leq n \leq 8$ . Then  $D = n$  holds generically. The preceding paragraph shows that there is  $u \in \mathbb{R}^n$  such that  $F(u)$  has a multiple eigenvalue which we can assume a maximal eigenvalue (because  $d = 3$ ). Then (4.2) proves that  $L(F)$  has an exposed face which is the image of a three-dimensional Euclidean ball under the map  $\mathcal{M}_3 \rightarrow L(F)$ . Thus,  $L(F) \subset \mathbb{R}^n$  has a large face unless  $F$  is equivalent modulo (1.1) to a tuple  $F^\varphi$  of Lemma 4.4. The set  $L(F^\varphi)$  is a Euclidean ball of dimension at most four. So  $D \geq 5$  suffices that  $L(F)$  has a large face and so  $L(F)$  has generically a large face when  $n \geq 5$ . For  $D = 4$  the Euclidean balls of Lemma 4.4 appear very special, which suggests that  $L(F)$  is generically no oval neither for  $n = 4$ .

The case of real symmetric 3-by-3 matrices  $F$  is an analogue to the hermitian case except for the fact that the dimension  $n$  for a level crossing is reduced by one as is the dimension of the exceptional Euclidean balls of Lemma 4.4.

We finish the section with an analysis of corner points of a subset  $C \subset \mathbb{R}^m$ , i.e. points with an  $m$ -dimensional normal cone. A set  $K \subset \mathbb{R}^m$  is a *convex cone* if  $K$  is convex and non-empty and if  $x \in K$ ,  $\lambda \geq 0$  implies  $\lambda x \in K$ . A point  $p \in C$  is called *conical point* [7] if  $C \subset p + K$  holds for a closed convex cone  $K \subset \mathbb{R}^m$  containing no line. Recall that the polar (definition above the 2nd proof of Fact 3.2) of a closed convex cone  $K \subset \mathbb{R}^m$  is a closed convex cone is given by  $K^\circ = \{u \in \mathbb{R}^m \mid \forall x \in K : \langle x, u \rangle \leq 0\}$ .

For any point  $p$  of a compact convex subset  $C \subset \mathbb{R}^m$  we have  $C \subset p + N(p)^\circ$  where  $N(p)$  is the normal cone of  $C$  at  $p$ , see e.g. Section 2.2 and equation (2.2) of [48]. Therefore, and since  $K \supset N(p)^\circ$  holds if and only if  $K^\circ \subset N(p)$ , we see that

$$(4.4) \quad K^\circ \subset N(p) \iff C \subset p + K.$$

Since  $K$  contains no line if and only if  $K^\circ$  has full dimension  $m$ , it follows for all  $p \in C$  that  $p$  is a corner point of  $C$  if and only if  $p$  is a conical point of  $C$ .

**Lemma 4.6.** *Let  $F$  be a sequence of  $n$  hermitian  $d$ -by- $d$  matrices,  $n, d \in \mathbb{N}$ . If  $p$  is a conical point, or equivalently a corner point, of  $L(F)$  then  $F$  is unitarily reducible. There exists a non-zero vector  $x \in \mathbb{C}^d$  such that  $F_i x = p_i x$  holds for  $i = 1, \dots, n$ .*

*Proof:* The equivalence of conical point and corner point is proved in the paragraph of (4.4). The remaining claims are proved in Proposition 2.5 of [7].  $\square$

We derive a classification of corner points for 3-by-3 matrices.

**Lemma 4.7.** *Let  $F$  be a sequence of  $n$  hermitian 3-by-3 matrices,  $n \in \mathbb{N}$ . If  $L(F)$  has a corner point  $p$  then  $D := \dim(L(F))$  satisfies  $D \in \{0, 1, 2, 3, 4\}$ . The sequence  $F$  is unitarily reducible, there exists a unitary  $U \in M_3$  such that  $U^* F U$  has the block-diagonal form  $U^* F U = ((p_1) \oplus G_1, \dots, (p_n) \oplus G_n)$  with  $G \in (M_2^h)^n$ . The convex support  $L(F)$  is (we ignore  $D = 0, 1$ ) the convex hull of  $p$  and of*

- ( $D = 2$ ) a segment whose affine hull does not contain  $p$  or an ellipse whose affine hull contains  $p$  but whose convex hull does not contain  $p$ ,
- ( $D = 3$ ) an ellipse whose affine hull does not contain  $p$  or an ellipsoid whose affine hull contains  $p$  but whose convex hull does not contain  $p$ ,
- ( $D = 4$ ) an ellipsoid whose affine hull does not contain  $p$ .

*Proof:* Lemma 4.6 proves that  $F$  is unitarily similar to the described block diagonal form. Thus, it is easy to prove that  $L(F)$  is the convex hull of  $p$  and  $L(G)$ . Since  $L(G)$  is a singleton, a segment, a filled ellipse, or a filled ellipsoid, only the cases listed above are possible.  $\square$

## 5. ARGUMENTS FOR THE CLASSIFICATION

We provide the arguments which we use in the introduction to prove the classification of the convex support  $L(F) \in \mathbb{R}^3$  of three hermitian 3-by-3 matrices  $F = (F_1, F_2, F_3)$ . Here we study intersections of large faces and we define a graph embedding.

A proper face of  $L(F)$  which is no singleton is by Lemma 4.2 an exposed face of the form of a segment or ellipse and is further a coatom of the lattice  $\mathcal{E}_{L(F)}$  of exposed faces. The set of these faces was denoted in (1.2) by  $\mathcal{L}(F)$ , they were called *large faces*. We will embed a graph into the union  $\bigcup_{G \in \mathcal{L}(F)} G$ . Lemma 5.1 and 5.4 rely on the classification of the numerical range of a 3-by-3 matrix [29, 30], namely on the fact that the numerical range of a 3-by-3 matrix with two one-dimensional faces is a triangle or the convex hull of an ellipse and a point outside the ellipse. We denote by  $P$  the projection  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ .

**Lemma 5.1.** *Let  $F \in (M_3^{\mathbb{h}})^3$ . If  $G_1, G_2 \in \mathcal{L}(F)$  then, after an affine transformation (1.1) acting on  $\mathbb{R}^3$ , the projected faces  $P(G_1)$  and  $P(G_2)$  are one-dimensional faces of  $L(F_1, F_2)$ . In particular  $G_1 \cap G_2 \neq \emptyset$  holds.*

*Proof:* We apply an affine transformation such that an outer normal vector of  $G_i$  is orthogonal to the  $x_3$ -axis,  $i = 1, 2$ . Since  $L(F_1, F_2) = P(L(F))$  holds, it follows that  $P(G_1)$  and  $P(G_2)$  are exposed faces of  $L(F_1, F_2)$ . If  $P(G_1) \cap P(G_2) \neq \emptyset$  then the claim  $G_1 \cap G_2 \neq \emptyset$  follows. A particular case is when neither  $P(G_1)$  nor  $P(G_2)$  is a singleton when the classification of the numerical range of a 3-by-3 matrix proves  $P(G_1) \cap P(G_2) \neq \emptyset$ .

On the contrary, let  $P(G_1) \cap P(G_2) = \emptyset$  with  $P(G_1) = \{x\}$  for an exposed point  $x \in L(F_1, F_2)$ . Then  $x$  has more than one pre-image under the linear map  $\mathcal{M}_3 \rightarrow L(F_1, F_2)$ , so Lemma 4.4 shows that  $L(F_1, F_2)$  is an ellipse and that  $x$  is the unique exposed point with more than one pre-image. All proper faces of the ellipse are singletons hence  $P(G_2) = \{y\}$  is a singleton, and by the same reasoning used for  $x$ , the exposed point  $y$  has more than one pre-image. This contradicts the uniqueness of an exposed point with more than one pre-image.  $\square$

Lemma 5.2 relies on studies of normal cones.

**Lemma 5.2.** *Let  $F \in (M_3^{\mathbb{h}})^3$  and let  $L(F)$  have no corner point. If  $G_1, G_2, G_3 \in \mathcal{L}(F)$  are mutually distinct then  $G_1 \cap G_2 \cap G_3 = \emptyset$ .*

*Proof:* The  $G_i$ 's are mutually distinct coatoms so  $G := G_1 \cap G_2 \cap G_3$  is strictly included in  $G_i$  for  $i = 1, 2, 3$ . The antitone isomorphism (3.1) shows that  $N(G_i)$

is strictly included in  $N(G)$ , Fact 3.2 shows that  $N(G_i)$  is a face of  $N(G)$ , and a separation theorem, see e.g. Theorem 1.3.2 of [48], shows  $\dim(N(G_i)) \geq 1$ . By contradiction we assume  $G \neq \emptyset$ . Then, since  $L(F)$  has no corner points,  $\dim(N(G)) \leq 2$  follows. This implies  $\dim(N(G_i)) = 1$  for  $i = 1, 2, 3$  and  $\dim(N(G)) = 2$ . But this is impossible, as a two-dimensional convex cone cannot have three one-dimensional faces. Therefore  $G = \emptyset$  holds.  $\square$

The existence of a corner point of  $L(F)$ , which is not allowed in Thm 5.3, yields the simple situation where  $F$  is unitarily reducible [7] and  $L(F)$  is one of the few objects of Lemma 4.7. In all other cases we embed a complete graph into the union of large faces having a vertex  $c(G)$  located at the centroid of every large face  $G \in \mathcal{L}(F)$ .

**Theorem 5.3** (Graph embedding). *Let  $F \in (M_3^h)^3$  and let  $L(F)$  have no corner point. Each two distinct faces  $G, H \in \mathcal{L}(F)$  intersect in a unique point  $p(G, H)$ . Define an edge between the vertices  $c(G)$  and  $c(H)$  as the union of the segment between  $c(G)$  and  $p(G, H)$  and the segment between  $p(G, H)$  and  $c(H)$ . Distinct edges of the graph intersect at most in a vertex of the graph.*

*Proof:* Lemma 5.1 shows that two given faces  $G, H \in \mathcal{L}(F)$  do intersect. Being faces of  $L(F)$  and ellipses or segments,  $G \neq H$  intersect in a unique extreme point  $p(G, H)$  of  $L(F)$ . By construction, for all  $G_i, H_i \in \mathcal{L}(F)$ ,  $G_i \neq H_i$ ,  $i = 1, 2$ , if the edge between  $c(G_1)$  and  $c(H_1)$  intersects the edge between  $c(G_2)$  and  $c(H_2)$  outside of a vertex then  $p(G_1, H_1) = p(G_2, H_2)$  follows. In that case Lemma 5.2 shows  $\{G_1, H_1\} = \{G_2, H_2\}$  which means that the two edges are equal.  $\square$

Unless  $L(F)$  has a corner point,  $\mathcal{L}(F)$  contains at most one segment.

**Lemma 5.4.** *Let  $F \in (M_3^h)^3$ . If  $\mathcal{L}(F)$  has two distinct faces which are segments then  $L(F)$  has a corner point.*

*Proof:* Let  $G, H \in \mathcal{L}(F)$  be distinct segments. Lemma 5.1 proves that, after an affine transformation,  $L(F_1, F_2)$  has two one-dimensional faces. Either  $L(F_1, F_2)$  is a triangle or the convex hull of an ellipse and a point outside of the ellipse.

In the first case, if  $L(F_1, F_2)$  is a triangle, another affine transformation allows us to assume that

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now  $L(F_1, F_2)$  is the convex hull of  $(0, -1)$ ,  $(0, 0)$ , and  $(-1, 0)$ . Equation (4.2) shows that the segment  $\mathbb{F}_{L(F)}(1, 0)$ , respectively  $\mathbb{F}_{L(F)}(0, 1)$ , is the image of a three-dimensional ball  $\mathcal{M}(p_1 M_3 p_1)$ , respectively  $\mathcal{M}(p_2 M_3 p_2)$ , where the diagonal matrix  $p_1 := \text{diag}(1, 0, 1)$ , respectively  $p_2 := (1, 1, 0)$ , is the spectral projection of  $F_i$  corresponding to the maximal eigenvalue zero,  $i = 1, 2$ . Let  $v := (1, 0, 0)$ ,  $v_1 := (0, 0, 1)$ , and  $v_2 := (0, 1, 0)$ . The matrix of  $p_1 F_2 p_1$  in the basis  $(v, v_1)$  is  $\text{diag}(0, -1)$ . Since  $\mathbb{F}_{L(F)}(1, 0)$  is a segment,  $p_1 F_3 p_1$  must be a linear combination of  $0 = p_1 F_1 p_1$  and of  $p_1 F_2 p_1$ . Therefore the off-diagonal terms  $\langle v, F_3 v_1 \rangle = (F_3)_{1,3}$  and  $\langle v_1, F_3 v \rangle = \overline{(F_3)_{1,3}}$ , must vanish. Analogously, since  $\mathbb{F}_{L(F)}(0, 1)$  is a segment we have  $(F_3)_{1,2} = (F_3)_{2,1} = 0$ . This proves that  $F$  is unitarily reducible. It is easy to see that  $(0, 0, (F_3)_{1,1})$  is a corner point of  $L(F)$  because  $L(F)$  is the convex hull of the point  $(0, 0, (F_3)_{1,1})$  and of  $L(\tilde{F})$  for the triple of 2-by-2 matrices  $\tilde{F}_{i,j} := F_{i+1,j+1}$ ,  $i, j = 1, 2$ , where  $L(\tilde{F})$  projects to the segment between  $(-1, 0)$  and  $(0, -1)$  in the  $x_1$ - $x_2$  plane.

Similarly as for a triangle we argue when  $L(F_1, F_2)$  is the convex hull of an ellipse and a point outside of the ellipse. Here, we have several affinely inequivalent cases. Lemma 5.1 of [49] proves that there is  $a > 1$  such that  $(F_1, F_2)$  is equivalent modulo (1.1) to  $(F'_1, F'_2)$ , where

$$F'_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad F'_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

An affine transformation then allows us to assume that  $F_1 = F'_1 + \sqrt{a^2 - 1}F'_2$  and  $F_2 = F'_1 - \sqrt{a^2 - 1}F'_2$  which have eigenvalues  $(-a, a, a)$ . The eigenvectors of  $F_1$ , respectively  $F_2$ , corresponding to  $a$  are  $v := (0, 0, 1)$  and

$$v_1 := ((1 - i\sqrt{a^2 - 1})/a, 1, 0), \quad \text{respectively} \quad v_2 := ((1 + i\sqrt{a^2 - 1})/a, 1, 0).$$

As in the first case, the off-diagonal entries  $\langle v_1, F_3 v \rangle$  and  $\langle v_2, F_3 v \rangle$  must vanish. So  $F$  is unitarily reducible and  $(a, a, (F_3)_{3,3})$  is a corner point of  $L(F)$ .  $\square$

## 6. EXAMPLES

We analyze and depict examples of convex support sets  $L(F)$  of three hermitian 3-by-3 matrices which have dimension three and no corner points, thereby proving that the corresponding eight classes of Table 1 are non-empty.

For all examples we compute the outer normal vectors  $u \in \mathbb{R}^3$  of proper exposed face of  $L(F)$  whose pre-image is not a singleton (we explain the method in Example 6.2 and just write formula and results later on). The question is left to the reader whether these faces are large faces and whether they are segments or ellipses. Indeed, the answer is a straightforward computation with 2-by-2 matrices (4.2): If  $F(u)$  has a maximal eigenvalue with spectral projection  $p$  then the exposed face  $\mathbb{F}_{L(F)}(u)$  is the convex support of the compressions of  $F_1, F_2, F_3$  to the range of  $p$ .

The graphics of Figures 2 and 3 were computed with Wolfram Mathematica using a heuristic method for which we have no proof of correctness: Recall from (1.3) the projective subvariety  $S_F$  of  $\mathbb{P}\mathbb{C}^4$ . The dual variety  $S_F^\wedge$  is a variety in  $\mathbb{P}\mathbb{C}^4$  which is the closure of the set of tangent hyperplanes to  $S_F$  at non-singular points [20, 18], a hyperplane  $\{(u_0 : u_1 : u_2 : u_3) \in \mathbb{P}\mathbb{C}^4 \mid u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0\}$  being identified with the point  $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}\mathbb{C}^4$ . The *boundary generating surface*  $S_F^\wedge(\mathbb{R})$  of  $F$  is [13] the real part of the affine part  $x_0 = 1$  of  $S_F^\wedge$ . We obtain an equation for  $S_F^\wedge(\mathbb{R})$  by computing a Groebner basis of the ideal generated by

$$p := \det(u_0 \mathbb{1} + u_1 F_1 + u_2 F_2 + u_3 F_3), \quad \partial_{u_i} p - x_i, \quad i = 0, 1, 2, 3,$$

when the variables  $(u_0, u_1, u_2, u_3)$  are eliminated ([14], Sec. 3.1). This ideal is a principal ideal in the following examples, generated by a polynomial  $\tilde{q}(x_0, x_1, x_2, x_3)$ . The boundary generating surface  $S_F^\wedge(\mathbb{R})$  is the zero set in  $\mathbb{R}^3$  of the polynomial

$$(6.1) \quad q(x_1, x_2, x_3) := \tilde{q}(1, x_1, x_2, x_3).$$

**Remark 6.1.** Notice that the polynomials  $\partial_{u_i} p - x_i$  are inhomogeneous and do not define a variety in the projective space  $\mathbb{P}\mathbb{C}^8$  or  $\mathbb{P}\mathbb{R}^8$ . Alternatively, Algorithm 5.1 of [46] uses the homogenous 2-by-2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ \partial_{u_0} p & \partial_{u_1} p & \partial_{u_2} p & \partial_{u_3} p \end{pmatrix}$$

and a suitable saturation ideal.

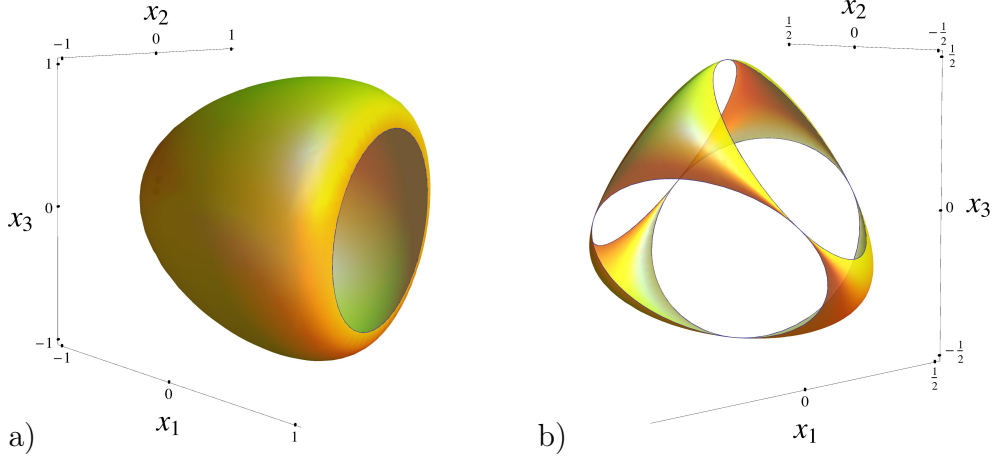


FIGURE 2. a) Object with one ellipse and b) object with four ellipses at the boundary. The depicted surfaces are the pieces of the boundary generating surfaces which lie on the boundary of the convex support.

The equation  $q(x_1, x_2, x_3) = 0$  of the boundary generating surface  $S_F^\wedge(\mathbb{R})$  is interesting in another respect. Recall for  $n = 2$  that the convex hull of  $S_F^\wedge(\mathbb{R})$  is the numerical range of  $F_1 + iF_2$  [29]. The analogue for  $n \geq 3$ , that the convex hull of  $S_F^\wedge(\mathbb{R})$  is the convex support  $L(F)$ , is wrong [13] because  $S_F^\wedge(\mathbb{R})$  can contain lines while  $L(F)$  is bounded. These lines are Zariski-closed proper subsets of  $S_F^\wedge(\mathbb{R})$  which in our examples is an irreducible surface. The phenomenon of non-constant dimension is well-known for real algebraic sets and cannot happen for complex algebraic sets, see Sec. 3.1 of [8]. We leave it to the reader to find lines in  $S_F^\wedge(\mathbb{R})$  by testing the coordinate axes to satisfy  $q(x_1, x_2, x_3) = 0$ .

Let us now discuss the examples. We skip the ovals and refer to Section 4 of [31] for examples of them.

**Example 6.2** (No segment, one ellipse). See Figure 1a) for a printout and Figure 2a) for a drawing.

$$F_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$q = -4x_1^3 - 4x_1^4 + 27x_2^2 + 18x_1x_2^2 - 13x_1^2x_2^2 - 32x_2^4 + 27x_3^2 + 18x_1x_3^2 - 13x_1^2x_3^2 - 64x_2^2x_3^2 - 32x_3^4.$$

The hermitian square in the sum of squares representation (4.3) of the discriminant of  $F(u)$ , corresponding to  $\nu := \{(1,1), (1,2), (1,3)\}$ , is

$$|M_\nu|^2 = (u_2^2 + u_3^2)^3/8.$$

Thus (4.2) proves that  $\mathbb{F}_{L(F)}(1, 0, 0)$  is the unique proper exposed face of  $L(F)$  whose pre-image under  $\mathbb{E}|_{\mathcal{M}_3}$  is no singleton. Lemma 4.2 shows that  $\mathbb{F}_{L(F)}(1, 0, 0)$  is in fact the unique proper face of  $L(F)$  with that property.

**Example 6.3** (No segment, two ellipses).

$$F_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$q = 4x_1x_2^2 - 4x_1^2x_2^2 - x_2^4 + 4x_3^2 - 4x_1^2x_3^2 - 4x_2^2x_3^2 - 4x_3^4.$$

The hermitian squares corresponding to  $\nu_1 := \{(1,1), (1,2), (3,3)\}$ ,  $\nu_2 := \{(1,1), (1,3), (2,2)\}$ , and  $\nu_3 := \{(1,1), (2,2), (3,3)\}$  are

$$\begin{aligned} |M_{\nu_1}|^2 &= (1 + u_1^2)^2, & \text{if } u_3 = 1, \\ |M_{\nu_2}|^2 &= u_2^2(u_2^2 - 2u_1^2)^2, \\ \text{and } |M_{\nu_3}|^2 &= u_1^2(u_2^2 - 2u_1^2)^2, & \text{if } u_3 = 0. \end{aligned}$$

Thus,  $\mathbb{F}_{L(F)}(-1, \pm\sqrt{2}, 0)$  are the unique proper faces of  $L(F)$  whose pre-images under  $\mathbb{E}|_{\mathcal{M}_3}$  are no singletons.

**Example 6.4** (No segment, three ellipses). See Figure 1b) for a printout of  $L(F)$  corresponding to the matrices (6.2). The first instance of an object without segment and with three ellipses which we found was

$$\begin{aligned} F_1 &:= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & F_2 &:= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & F_3 &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ q &= -4x_1^6 - 24x_1^4x_2^2 + 27x_2^4 - 48x_1^2x_2^4 - 32x_2^6 + 36x_1^2x_2^2x_3 + 18x_2^4x_3 + 8x_1^4x_3^2 \\ &\quad - 4x_1^2x_2^2x_3^2 - 13x_2^4x_3^2 - 4x_2^2x_3^3 - 4x_1^2x_3^4 - 4x_2^2x_3^4. \end{aligned}$$

The hermitian squares corresponding to  $\nu_1 := \{(1,1), (1,2), (2,2)\}$  and  $\nu_2 := \{(1,1), (1,2), (3,3)\}$  are

$$\begin{aligned} |M_{\nu_1}|^2 &= (1 + 2u_1^2)/8, & \text{if } u_2 = 1, \\ \text{and } |M_{\nu_2}|^2 &= u_1^2(u_1^2 - 4u_3^2)^2, & \text{if } u_2 = 0. \end{aligned}$$

Thus,  $\mathbb{F}_{L(F)}(0, 0, 1)$ , and  $\mathbb{F}_{L(F)}(\pm 2, 0, -1)$  are the unique proper faces of  $L(F)$  whose pre-images under  $\mathbb{E}|_{\mathcal{M}_3}$  are no singletons.

Out of curiosity we mention also the example

$$(6.2) \quad F_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_3 := \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

where  $L(F)$  also has three ellipses and no segments as proper faces. Here the normal vectors of the ellipses are mutually orthogonal and  $q(x_1, x_2, x_3)$  is a degree six polynomial with the maximal number of 84 monomials.

**Example 6.5** (No segment, four ellipses). This example has appeared in [26]. See Figure 1c) for a printout and Figure 2b) for a drawing.

$$\begin{aligned} F_1 &:= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & F_2 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & F_3 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ q &= x_1x_2x_3 - x_1^2x_2^2 - x_1^2x_3^2 - x_2^2x_3^2. \end{aligned}$$

The surface  $\{x \in \mathbb{R}^3 \mid q(x) = 0\}$  is also known as the Roman surface. For all unit vectors  $u \in \mathbb{R}^3$  the discriminant of  $F(u)$  is

$$\begin{aligned} \delta(F(u)) &= \frac{1}{32}((u_1^2 - u_2^2)^2 + (u_2^2 - u_3^2)^2 + (u_3^2 - u_1^2)^2 \\ &\quad + 6(u_3^2(u_1^2 - u_2^2)^2 + u_1^2(u_2^2 - u_3^2)^2 + u_2^2(u_3^2 - u_1^2)^2)). \end{aligned}$$

Thus (4.2) proves that

$$\begin{aligned} \mathbb{F}_{L(F)}(-1, -1, -1), & & \mathbb{F}_{L(F)}(-1, 1, 1), \\ \mathbb{F}_{L(F)}(1, -1, 1), & & \text{and } \mathbb{F}_{L(F)}(1, 1, -1) \end{aligned}$$

are the unique proper faces of  $L(F)$  whose pre-images under  $\mathbb{E}|_{\mathcal{M}_3}$  are no singletons.



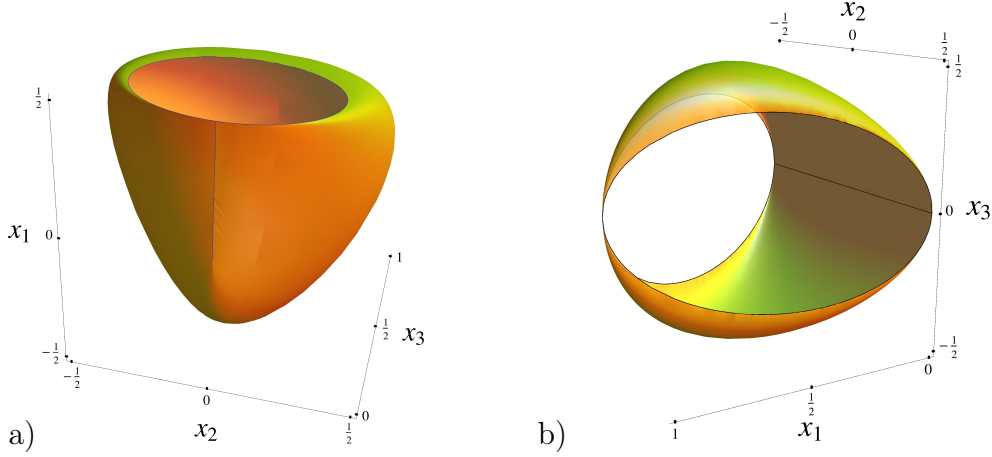


FIGURE 3. a) Object with one segment and one ellipse and b) object with one segment and two ellipses. The depicted surfaces are the pieces of the boundary generating surfaces which lie on the boundary of the convex support.

**Example 6.6** (One segment, no ellipses). See Figure 1d) for a printout.

$$F_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & 1 & 0 \end{pmatrix},$$

the polynomial  $q$  has degree eight and 31 monomials. The hermitian squares corresponding to  $\nu_1 := \{(1,1), (1,2), (1,3)\}$  and  $\nu_2 := \{(1,1), (1,2), (3,3)\}$  are

$$\begin{aligned} |M_{\nu_1}|^2 &= (1 + 4u_1^4 + 4u_1^2(1 + 4u_2^2))/8, & \text{if } u_3 = 1, \\ \text{and } |M_{\nu_2}|^2 &= u_1^6, & \text{if } u_3 = 0. \end{aligned}$$

Thus,  $\mathbb{F}_{L(F)}(0, 1, 0)$  is the unique proper face of  $L(F)$  whose pre-image under  $\mathbb{E}|_{\mathcal{M}_3}$  is no singleton.

**Example 6.7** (One segment, one ellipse). This object has appeared in [13]. Let

$$F_1 := \frac{1}{2} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_2 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$q = -4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3\lambda - x_2^4\lambda^2.$$

Unitary conjugation  $u^*F_1u$ ,  $u^*F_2u$ ,  $u^*F_3u$  with the diagonal matrix  $\text{diag}(i, 1, 1)$  gives matrices whose convex support is, for  $\lambda = 2$ , affinely isomorphic to the example in Section 3 of [13]. See Figure 1e) for a printout. A drawing at  $\lambda = 1$  is depicted in Figure 3a).

Lemma 4.4 shows for  $\lambda = 0$  that  $L(F)$  is the ball of radius  $\frac{1}{2}$  centered at  $(0, 0, \frac{1}{2})$  and the origin is the unique boundary point whose pre-image is not a singleton.

For  $\lambda = 1$  the hermitian squares corresponding to  $\nu_1 := \{(1,1), (1,2), (1,3)\}$ ,  $\nu_2 := \{(1,1), (1,3), (2,2)\}$ , and  $\nu_3 := \{(1,1), (2,2), (2,3)\}$  are

$$\begin{aligned} |M_{\nu_1}|^2 &= u_1^2/64, & \text{if } u_2 = 1, \\ |M_{\nu_2}|^2 &= 1/64, & \text{if } u_2 = 1, u_1 = 0, \\ \text{and } |M_{\nu_3}|^2 &= u_1^4u_3^2/16, & \text{if } u_2 = 0. \end{aligned}$$

Thus,  $\mathbb{F}_{L(F)}(1, 0, 0)$  and  $\mathbb{F}_{L(F)}(0, 0, -1)$  are the unique proper faces of  $L(F)$  whose pre-images under  $\mathbb{E}|_{\mathcal{M}_3}$  are no singletons.

**Example 6.8** (One segment, two ellipses). This object has appeared in [9]. Let

$$F_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_2 := \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$q = -x_1^2 x_2^2 + x_1 x_3^2 - x_1^2 x_3^2 - x_3^4.$$

Othogonal conjugation with the matrix  $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} / \sqrt{2}$  produces matrices whose convex support is affinely isomorphic to Example 6 of [9]. See Figure 1f) for a printout and Figure 3b) for a drawing. The hermitian squares corresponding to  $\nu_1 := \{(1,1), (1,3), (2,2)\}$  and  $\nu_2 := \{(1,1), (1,2), (3,3)\}$  are

$$\begin{aligned} |M_{\nu_1}|^2 &= 1/64, & \text{if } u_3 = 1, \\ \text{and } |M_{\nu_2}|^2 &= u_2^2(u_2^2 - 4u_1^2)^2/64, & \text{if } u_3 = 0. \end{aligned}$$

Thus,  $\mathbb{F}_{L(F)}(-1, 0, 0)$  and  $\mathbb{F}_{L(F)}(1, \pm 2, 0)$  are the unique proper exposed faces of  $L(F)$  whose pre-images under  $\mathbb{E}|_{\mathcal{M}_3}$  are no singletons. It is easy to find examples with one segment and two ellipses whose outer normal vectors are not coplanar.

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